SIMPLICITY OF THE LIE ALGEBRA OF SKEW SYMMETRIC ELEMENTS OF A LEAVITT PATH ALGEBRA

ADEL ALAHMEDI AND HAMED ALSULAMI

Abstract. For a field F of characteristic not 2 and a directed row-finite graph Γ let $L(\Gamma)$ be the Leavitt path algebra with the standard involution $*$. We study the Lie algebra $K = K(L(\Gamma), *)$ of $*-$skew-symmetric elements and find necessary and sufficient conditions for the Lie algebra $[K, K]$ to be simple.

1. Introduction.

G. Abrams and G. Aranda Pino in [1] have shown that a Leavitt path algebra of a graph $\Gamma(V, E)$ is simple if and only if (i) the set of vertices V has no proper hereditary and saturated subsets (ii) every cycle in Γ has an exit. In [2] G. Abrams and Z. Mesyan found necessary and sufficient conditions for the Lie algebra $[L(\Gamma), L(\Gamma)]$ to be simple under the assumption that $L(\Gamma)$ is simple. They proved that if V is infinite then $[L(\Gamma), L(\Gamma)]$ is simple. If V is finite then $[L(\Gamma), L(\Gamma)]$ is simple if and only if $1_{L(\Gamma)} = \sum_{v\in V} \notin [L(\Gamma), L(\Gamma)]$. In this paper we study the Lie algebra $K = K(L(\Gamma), *)$ of $*-$skew-symmetric elements and prove that $[K, K]$ is simple if and only if the graph Γ is almost simple, see the Definition 4 below.

2. Definitions and Terminology

A (directed) graph $\Gamma = (V, E, s, r)$ consists of two sets V and E that are respectively called vertices and edges, and two maps s, $r : E \to V$.The vertices $s(e)$ and $r(e)$ are referred to as the source and the range of the edge e, respectively. The graph is called row-finite if for all vertices $v \in V$,$card(s^{-1}(v)) < \infty$. A vertex v for which $(s^{-1}(v)) = \emptyset$ is called a sink. A vertex v such that $r^{-1}(v) = \emptyset$ is called a source. A path $p = e_1.....e_n$ in a graph Γ is a sequence of edges $e_1.....e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, ..., n-1$. In this case we say that the path p starts at the vertex $s(e_1)$ and ends at the vertex $r(e_n)$. If $s(e_1) = r(e_n)$, then the path is closed. If $p = e_1.....e_n$ is a closed path and the vertices $s(e_1)$, ..., $s(e_n)$ are distinct, then the subgraph ($s(e_1), ..., s(e_n); e_1, ..., e_n$) of the graph Γ is called a cycle. A cycle of length 1 is called a loop.

Definition 1. We call an edge $e \in E$ a *fiber* if $s(e)$ is source, $r(e)$ is sink and $E(V, r(e)) = \{e\}$.

Definition 2. We call a connected graph Γ a *fork* if one vertex in V is a source, whereas all other vertices are sinks.

Key words and phrases. Leavitt Path Algebra. Cuntz-Krieger C*-Algebras. Simple Lie Algebra.

Definition 3. We call a vertex v in a connected graph $\Gamma(V,E)$ a *balloon* over a nonempty subset W of V if (i) $v \notin W$, (ii) there is a loop $C \in E(v,v)$, (iii) $E(v,W) \neq \emptyset$, (iv) $E(v,V) = \{C\} \cup E(v,W)$, and (v) $E(V,v) = \{C\}$.

Let Γ be a row-finite graph and let F be a field. The Leavitt path F-algebra $L(\Gamma)$ is the F-algebra presented by the set of generators $\{v|v \in V\}$, $\{e, e^*|\ e \in E\}$ and the set of relators (1) $v_iv_j = \delta_{v_i,v_j}v_i$ for all $v_i, v_j \in V$; (2) $s(e)e = er(e) = e$, $r(e)e^* = e^*s(e) = e^*$ for all $e \in E$; (3) $e^*f = \delta_{e,f}r(e)$, for all $e,\ f \in E$; (4) $v = \sum\limits_{s(e)=v} ee^*$, for an arbitrary vertex v which is not a sink. The mapping which sends v to v for $v \in V$, e to e^* and e^* to e for $e \in E$, extends to an involution of the algebra $L(\Gamma)$. If $p = e_1.....e_n$ is a path, then $p^* = e_n^*....e_1^*$. In what follows we consider only row-finite directed graphs. We call a graph Γ simple if the Leavitt path algebra $L(\Gamma)$ is simple.

Definition 4. We call a graph Γ almost simple if Γ contains subgraphs $\Gamma \supset \Gamma_1 \supset \Gamma_2$ where Γ_2 is a simple; Γ_1 is obtained from Γ_2 by adding balloons; Γ is obtained from Γ_1 by adding fibers.

Let A be an associative $F-$algebra. For elements $a, b \in A$, let $[a,b] = ab - ba$ be their the commutator. Then $A^{(-)} = (A, [,])$ is a Lie algebra. The space of $*-$skew-symmetric elements $K = K(L(\Gamma), *) = \{a \in A | a^* = -a\}$ is a subalgebra of the Lie algebra $L(\Gamma)^{(-)}$.

Theorem 2.1. *The Lie algebra $[K,K]$ is simple if and only if the graph Γ is almost simple.*

3. Almost Simple Graph

Lemma 1. *Let $\Gamma(V,E)$ be a directed graph.*

(i) *If e, f are different edges and $r(e) = s(f)$, then $[e - e^*, f - f^*] \neq 0$.*
(ii) *If e, f are different edges and $r(e) = r(f)$, then $[e - e^*, f - f^*] \neq 0$.*

Proof. (i) Since e, f are different edges we have $ef^* = fe^* = 0$. Hence $[e - e^*, f - f^*] = ef - ef^* + (fe)^* - fe + fe^* - (ef)^*$. The nonzero elements among ef, ef^*, $(fe)^*$, fe, fe^*, $(ef)^*$ are linearly independent: for example they can be included in a base of [3, Thm 1]. The element ef is nonzero. This proves the assertion.

(ii) Instead of the nonzero element ef in the argument above we use the fact that $ef^* \neq 0$, $fe^* \neq 0$.

□

Lemma 2. *If $[K,K] = 0$, then Γ is a disjoint unions of points, loops and forks.*

Proof. This lemma immediately follows from Lemma 1. □

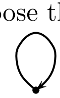
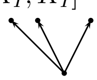
Recall that a nonempty subset $W \subseteq V$ is said to be hereditary if $r(s^{-1}(w)) \subseteq W$ for every vertex $w \in W$, see [1]. A nonempty subset $W \subseteq V$ is saturated if every non-sink $v \in V$, such that $r(s^{-1}(v)) \subseteq W$, lies in W, see [1]. If W is a hereditary and saturated subset of V, then the ideal I generated by W in $L(\Gamma)$ is spanned by all elements pq^*, where p, q are paths, $r(p) = r(q) \in W$. In this case $L(\Gamma)/I \cong L(\Gamma')$, where $\Gamma' = (V \setminus W, E \setminus E(V,W))$, see [1].

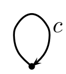
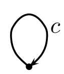
Lemma 3. *Let $\Gamma(V, E)$ be a connected graph, $card(E) > 1$, and $e \in E$ be a fiber. Let $\Gamma' = (V \setminus \{r(e)\}, E \setminus \{e\})$. Then $L(\Gamma) \cong L(\Gamma') \oplus M_2(F)$.*

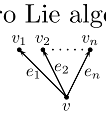
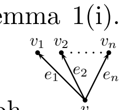
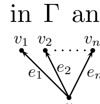
Proof. If $s^{-1}(s(e)) = \{e\}$ then $s(e) \xrightarrow{e} r(e)$ is an isolated subgraph, hence it is equal to Γ, which contradicts our assumption. Hence $s^{-1}(s(e)) \supsetneqq \{e\}$. This implies that $\{r(e)\}$ is a hereditary and saturated subset of V. The ideal $I = id_{L(\Gamma)}(r(e))$ is the $F-$span of $\{pq^* : r(p) = r(q) = r(e)\}$. Looking at passible pathes we see that $I = Fe + Fe^* + Fr(e) + Fee^* \cong M_2(F)$. Since every ideal with an identity is a direct summand, we have $L(\Gamma) \cong L(\Gamma)/I \oplus M_2(F) \cong L(\Gamma') \oplus M_2(F)$. □

Corollary 1. *Let Γ be a connected graph, which is not a fork. Let $E_1 = \{e \in E | e$ is a fiber $\}$, $\Gamma' = (V \setminus r(E_1), E \setminus E_1)$. Then $L(\Gamma) \cong L(\Gamma') \oplus A$, where A is a direct sum of $card(E_1)$ copies of $M_2(F)$. In particular, $[K(L(\Gamma), *), K(L(\Gamma), *)] \cong [K(L(\Gamma'), *), K(L(\Gamma'), *)]$.*

Lemma 4. *Let $\Gamma(V, E)$ be a fiber-less graph. Let W be a nonempty hereditary and saturated subset of V. Let $I = id_{L(\Gamma)}(W)$ and let $K_I = K(I, *)$. If $[K, K]$ is simple then $[K_I, K_I] \neq (0)$.*

Proof. Suppose that $[K_I, K_I] = (0)$. Then by Lemma 2 $(W, E(W, W))$ is a disjoint union of $\bullet$, , and .

Suppose that the graph $(W, E(W, W))$ contains a loop  v. Then $E(v, V) = E(v, W) = \{c\}$. If $e \in E(V, v)$ and $e \neq c$, then both $e, c \in I$ and $[e - e^*, c - c^*] \neq 0$, by Lemma 1(ii), which contradicts our assumption. Hence $E(V, v) = \{c\}$. Thus v is isolated in Γ, and hence $\Gamma =$  v. But our assumption was that $[K, K]$ is a simple nonzero Lie algebra, a contradiction.

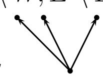
Let  be a part of $(W, E(W, W))$. Now, $E(v_i, V) = E(v_i, W)$ for $i = 1, 2, \ldots, n$, so $v_1, \ldots, v_n$ are sinks. If $e \in E(V, v_i)$ and $e \neq e_i$, then $e, e_i \in I$, $[e - e^*, e_i - e_i^*] \neq 0$, a contradiction. Hence $E(V, v_i) = \{e_i\}$. Similarly if $e \in E(V, v)$, then $e \in I$, $[e - e^*, e_1 - e_1^*] \neq 0$ by Lemma 1(i). Hence v is a source in Γ and $E(v, V) = \{e_1, \ldots, e_n\}$. Thus the subgraph  is isolated in Γ, so $\Gamma =$ . Hence $L(\Gamma) \cong \oplus_1^n M_2(F)$ and hence $[K, K] = 0$. Contradiction.

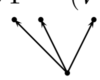
Let $\overset{\bullet}{v}$ be a part of $(W, E(W, W))$. Then v is a sink in Γ. Suppose that $v_1 \in V \setminus W$, and $e \in E$ with $s(e) = v_1$ and $r(e) = v$.

If $f \in E(V, v)$ and $e \neq f$, then $e, f \in I$ and $[e - e^*, f - f^*] \neq 0$ by Lemma 1(ii), a contradiction. Hence $E(V, v) = \{e\}$. If $g \in E(V, v_1)$, then $ge \in I$ and $[ge - e^* g^*, e - e^*] = gee^* - ee^* g^* \neq 0$, a contradiction. Hence v_1 is a source. Thus e is a fiber. But we assumed that Γ is fiber-less, a contradiction. Thus $[K_I, K_I] \neq (0)$. □

Lemma 5. *Let $\Gamma(V,E)$ be a fiber-less graph. Let W be a nonempty hereditary and saturated subset of V. Let $I = id_{L(\Gamma)}(W)$ and let $K_I = K(I,*)$. Let $\Gamma' = (V\backslash W, E\backslash E(V\backslash W))$. Let $K' = K(L(\Gamma'),*)$. If $[K,K]$ is simple, then $[K',K'] = (0)$.*

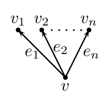
Proof. Since $L(\Gamma') \cong L(\Gamma)/I$ and by Lemma 4 $[K_I, K_I]$ is a nonzero ideal in $[K,K]$ which is simple, it follows that $[K,K] = [K_I,K_I] \subseteq I$. Hence $[K',K'] = (0)$. $\square$

Corollary 2. *Let $\Gamma(V,E)$ be a fiber-less graph. Let W be nonempty hereditary and saturated subset of V. Let $I = id_{L(\Gamma)}(W)$ and let $K_I = K(I,*)$. Let $\Gamma' = (V\setminus W, E\setminus E(V\setminus W))$. If $[K,K]$ is simple, then Γ' is a disjoint union of* ,  , *and*  .

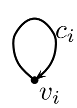
Lemma 6. *Let $\Gamma(V,E)$ be a fiber-less graph. Let W be nonempty hereditary and saturated subset of V. Let $\Gamma' = (V\setminus W, E\setminus E(V\setminus W))$. If $[K,K]$ is simple, then Γ' does not contain* *and*  .

Proof. Let 

be a component of Γ'. Then $E(v, V\setminus W) = \emptyset$. If $s^{-1}(v) \neq \emptyset$, then $r(s^{-1}(v)) \subset W$ and since W is saturated we will have $v \in W$. Hence v is a sink. Since v is isolated in Γ', it follows that $E(V\setminus W, v) = \emptyset$. Also since $v \notin W$ and $E(V\setminus W, v) = \emptyset$, then $E(W,v) = \emptyset$. Hence $E(V,v) = \emptyset$ and hence v is isolated in Γ, a contradiction.

Now suppose that 

is a component of Γ'. Since v_i, for $i = 1,2,\ldots,n$, has no immediate descendants in $V\setminus W$ it follows that $E(v_i, V) = E(v_i, W)$. Since $v_i \notin W$ we conclude that each v_i is a sink. If e is another edge different from e_i arriving at v_i, then $s(e)$ must lie in W and hence v_i belongs to W, which is a contradiction. Now suppose that an edge e arrives at v. Again $s(e)$ can not be in W and can not be in $V\setminus W$. Hence v is a source in Γ. But then $e_1,\ldots,e_n$ are fibers. A contradiction. $\square$

Lemma 7. *Let $\Gamma(V,E)$ be a fiber-less graph. Let W be a nonempty hereditary and saturated subset of V. Let $\Gamma' = (V\setminus W, E\setminus E(V\setminus W))$. Suppose that $[K,K]$ is simple, and Γ' is a disjoint union of loops* 

,$i \in \Omega$. Then v_i, $i \in \Omega$ are balloons over W.

Proof. Since $E(W,v_i) = \emptyset$ and $E(V\setminus W, v_i) = \{c_i\}$ as in Γ' it follows that $E(V,v_i) = \{c_i\}$. Moreover, $E(v_i, V\setminus W) = \{c_i\}$ as in Γ'. If $E(v_i,W) = \emptyset$, then v_i is isolated in Γ, hence $E(v_i,W) \neq \emptyset$. Thus v_i is a balloon over W. $\square$

Lemma 8. *Let $d \geq 1$ be a positive integer. Then the Lie algebra $[K(M_d(F[t,t^{-1}]),*), K(M_d(F[t,t^{-1}]),*)]$ is not simple.*

Proof. Note that the standard involution on $K(M_d(F[t,t^{-1}]))$ is $(f_{ij}(t))^* = \left(f_{ji}(t^{-1})\right)$. For $d=1$, $[K(F[t,t^{-1}]), K(F[t,t^{-1}])] = 0$. Let $d \geq 2$. Let J_n be the ideal of $F[t,t^{-1}]$ generated by $(1-t)^n$. Since $((1-t)^n)^* = ((1-t)^*)^n = (1-t^{-1})^n = (-1)^n t^{-n}(1-t)^n$, it follows that $J^* = J$. We will show that $[K(M_d(J_n),*), K(M_d(J_n),*)] \neq (0)$ for all $n \geq 1$. It is sufficient to do it for $d=2$. For $f, g \in F[t,t^{-1}]$, we have

$$\left[\begin{pmatrix} 0 & f(t) \\ -f(t^{-1}) & 0 \end{pmatrix}, \begin{pmatrix} 0 & g(t) \\ -g(t^{-1}) & 0 \end{pmatrix}\right] = \begin{pmatrix} g(t)f(t^{-1}) - f(t)g(t^{-1}) & 0 \\ 0 & f(t)g(t^{-1}) - f(t^{-1})g(t) \end{pmatrix}$$

Let $f(t) = (1-t)^n$ and $g(t) = (1-t)^m$ where $n < m$. Then $f, g \in J_n$ and the matrix on the right hand side is nonzero. If $[K(M_d(F[t,t^{-1}]),*), K(M_d(F[t,t^{-1}]),*)]$ is simple, then it is equal to $[K(M_d(J_n),*), K(M_d(J_n),*)]$ and therefore lies in $M_d(J_n)$. But $\bigcap\limits_n J_n = (0)$, a contradiction.

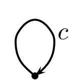
Hence $[K(M_d(F[t,t^{-1}]),*), K(M_d(F[t,t^{-1}]),*)]$ is not simple. □

Lemma 9. *Let $\Gamma(V,E)$ be a fiber-less graph. If $[K,K]$ is simple, then every cycle in Γ has an exit or is a loop.*

Proof. Let C be a cycle, which does not have an exit. Let $a = \sum\limits_i v_i$ be the sum of all vertices on C. Then a is an idempotent and $aL(\Gamma)a = L(C) \cong M_d(F[t,t^{-1}])$, where d is the number of vertices on C. For an arbitrary ideal $I \lhd L(C)$ denote $\widetilde{I} = id_{L(\Gamma)}(I)$. Then $a\widetilde{I}a \subseteq I$. If $I^* = I$, then $\widetilde{I}^* = \widetilde{I}$. Let J_n be the ideal of $F[t,t^{-1}]$ generated by $(1-t)^n$. Let $I_n = id_{L(\Gamma)}(M_d(J_n))$. As shown in the proof of Lemma 8, $[K(I_n,*), K(I_n,*)] \neq (0)$. Since $[K,K]$ is simple it follows that $[K,K] = [K(I_n,*), K(I_n,*)]$. In particular, $[K,K] \cap L(C) \subseteq M_d(J_n)$. Since $\cap_n J_n = (0)$ we conclude that $[K,K] \cap L(C) = (0)$. By Lemma 8 we have $(0) \neq [K(M_d(F[t,t^{-1}]),*), K(M_d(F[t,t^{-1}]),*)] \subseteq M_d(J_n)$ for $d \geq 2$. Hence $d=1$. Thus C is a loop. If there is no edge arriving at C from outside , then C is isolated in Γ and hence $\Gamma = C$, which is a contradiction, since $[K,K]$ is simple. Hence there is an edge e such that $s(e) \notin C$ and $r(e) = v \in V(C)$. Let J_n be the ideal generated by $(v-C)^n$ in $L(C)$. Let $I_n = id_{L(\Gamma)}(J_n)$. Let $f, g \in J_n$. Now, $ef, eg \in I_n$. We have $[ef - (ef)^*, eg - (eg)^*] = -efg^*e^* - f^*g + egf^*e^* + gf^* \neq 0$. Now, $[K,K] = [K(I_n,*), K(I_n,*)]$ and $v[K,K]v \subseteq J_n$. Since $\cap_n J_n = (0)$ it follows that $v[K,K]v = (0)$. But $v[ef-(ef)^*, eg-(eg)^*]v = g^*f - f^*g \neq 0$ for some $f, g \in L(C)$, a contradiction. Hence C has an exit.

□

Let $\Gamma(V,E)$ be a fiber-less connected graph such that the Lie algebra $[K(L(\Gamma),*), K(L(\Gamma),*)]$ is simple. Let W_1, W_2 be two nonempty hereditary saturated subsets of V such that $W_1 \cap W_2 = \emptyset$. If $w \in W_2$, then by Lemma7 w is a balloon over W_1. It means that there exist a loop

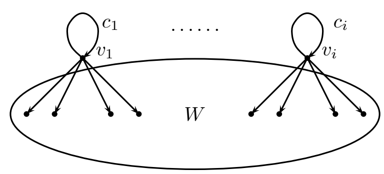


, $r^{-1}(w) = \{c\}$, all descendants of w except w itself lie in W_2. Since W_2 is hereditary and $W_1 \cap W_2 = \emptyset$ it follows that the point w is isolated in Γ. Since Γ is connected, $V = \{w\}$, a contradiction. We showed that any two nonempty hereditary saturated subsets of V intersect. Now let $\{W_i\}$ be the family of all nonempty hereditary saturated subsets of V. We claim that $\cap_i W_i \neq \emptyset$. Indeed, choose a vertex $v \in V$. If $\cap_i W_i = \emptyset$ then there exists a

subset W_i such that $r(s^{-1}(v)) \cap W_i = \emptyset$. By Lemma7 v is a balloon over W_i. Since W_i does not contain any range of an edge originating at v we conclude, as above, that the point v is isolated in V, a contradiction. Thus $W = \cap_i W_i$ is the smallest nonempty hereditary saturated subset of V. Since every hereditary and saturated subset of W is a hereditary and saturated subset of V it follows that W does not contains proper hereditary and saturated subsets. Now, if $[K, K]$ is simple then by this and Lemma 9 we see that $\Gamma_W = (W, E(W, W))$ satisfies the conditions of [1]. Hence $L(\Gamma_W)$ is a simple algebra. Hence we proved Theorem in one direction: if the Lie algebra$[K, K]$ is simple then the graph Γ is almost simple.

4. Simplicity of the Lie algebra $[K, K]$.

Let $\Gamma(V, E)$ be a graph. Suppose that W is a nonempty subset of V, each vertex from $V \setminus W$is a balloon over W and the Leavitt path algebra $L(W, E(W, W))$ is simple. Our aim is to show that the Lie algebra $[K(L(\Gamma), *), K(L(\Gamma), *)]$ is simple.



The subset W in V is hereditary and saturated. Let $I = id_{L(\Gamma)}(W)$. Then $L(\Gamma)/I$ is a direct sum of $card(V \setminus W)$- many copies of the algebra $F[t, t^{-1}]$. Denote $E_i = E(v_i, W)$, $v_i \in V \setminus W$, c_i is a loop from $E(v_i, v_i)$. Then $I = L(W) + \sum_i c_i^{k_i} E_i L(W) + \sum_j L(W) E_j^* (c_j^*)^{r_j} + \sum_{i,j} c_i^{k_i} E_i L(W) E_j^* (c_j^*)^{r_j}$ This decomposition is related to the family of pairwise orthogonal idempotents $\{u = \sum_{v \in W} v\} \dot{\bigcup} (\cup_i \mathcal{E}_i)$, $\mathcal{E}_i = \{c_i^k e e^* (c_i^*)^k | k \geq 0, e \in E_i\}$.

Lemma 10. *I is a simple algebra.*

Proof. We have $I = uIu + \sum_i uI\mathcal{E}_i + \sum_i \mathcal{E}_i I u + \sum_{i,j} \mathcal{E}_i I \mathcal{E}_j$. Also, we have $I = L(W) + \sum_i c_i^{k_i} E_i L(W) + \sum_j L(W) E_j^* (c_j^*)^{r_j} + \sum_{i,j} c_i^{k_i} E_i L(W) E_j^* (c_j^*)^{r_j}$. Let J be a nonzero ideal in I. Suppose that $\mathcal{E}_i J \mathcal{E}_j \neq (0)$. Thus there exist $e \in \mathcal{E}_i, f \in \mathcal{E}_j$, $k_i, k_j \geq 0$ such that $(0) \neq c_i^{k_i} e e^* (c_i^*)^{k_i} J c_j^{k_j} f f^* (c_j^*)^{k_j}$. This subspace lies in $c_i^{k_i} e L(W) f^* (c_j^*)^{k_j} \cap J$. Multiplying on the left by $e (c_i^*)^{k_i} \in I$ and on the right by $c_j^{k_j} f \in I$, we get $L(W) \cap J \neq 0$. This and the simplicity of $L(W)$ implies $L(W) \subseteq J$ and therefore $J = I$. Suppose that $uJ\mathcal{E}_j \neq (0)$. Then there exist an edge $e \in E_j$ and $k \geq 0$ such that $(0) \neq uJc_j^k e e^* (c_j^*)^k \subseteq L(W) e^* (c_j^*)^k \cap J$. Multiplying on the right by $c_j^k e \in I$, we get again $L(W) \cap J \neq (0)$ and hence $I = J$. The case $\mathcal{E}_i J u \neq (0)$ is treated similarly. If $\mathcal{E}_i J u = uJ\mathcal{E}_j = \mathcal{E}_i J \mathcal{E}_j = (0)$ for all i, j then $J \subseteq L(W)$ and hence $I = J$.

$\square$

Lemma 11. *Let A be an arbitrary simple associative algebra with an involution $*: A \to A$ and three pairwise orthogonal symmetric idempotents e_1, e_2, e_3. Then $K(A,*) = e_i K(A,*) e_i + [K(A,*), K(A,*)]$ for $i = 1,2,3$.*

Proof. Denote $K = K(A,*)$. For an arbitrary element $a \in A$ denote $\{a\} = a - a^*$. For distinct $i,j,k \in \{1,2,3\}$ and for arbitrary elements $a,b \in A$ we have $\{e_i a e_j b e_k\} = [e_i a e_j - e_j a^* e_i, e_j b e_k - e_k b^* e_j] \in [K,K]$. Since $A = Ae_jA$ it follows that $\{e_i A e_k\} = \{e_i A e_j A e_k\} \subseteq [K,K]$. For arbitrary elements $a,b \in A$ we have $[ae_1 - e_1 a^*, e_1 b - b^* e_1] = \{ae_1 b\} - \{ae_1 b^* e_1\} - \{e_1 a^* e_1 b\} + \{e_1 a^* b^* e_1\} \in [K,K]$. Hence $\{ae_1 b\} \in \{e_1 A\} + [K,K]$. Since $A = Ae_1A$ it implies that $K = \{e_1 A\} + [K,K]$. Again, for arbitrary elements $a,b \in A$ we have $[e_1 a e_2 - e_2 a^* e_1, e_2 b - b^* e_2] = \{e_1 a e_2 b\} - \{e_1 a e_2 b^* e_2\} + \{e_2 a^* e_1 b^* e_2\} \in [K,K]$. Hence $\{e_1 a e_2 b\} \in \{e_1 A e_2\} + e_2 K e_2 + [K,K] = e_2 K e_2 + [K,K]$. Since $A = Ae_2A$ we get $\{e_1 A\} \subseteq e_2 K e_2 + [K,K]$ and in view of the earlier inclusions, $K = e_2 K e_2 + [K,K]$. Similarly we can show that $K = e_i K e_i + [K,K]$ for $i = 1,2,3$. □

Lemma 12. *$[K(L(\Gamma),*), K(L(\Gamma),*)] = [K(I,*), K(I,*)]$. In particular, this Lie algebra is simple.*

Proof. As above Let $V \setminus W = \{v_i | i \in \Omega\}$, $E(v_i, v_i) = \{c_i\}$. Then $K(L(\Gamma),*) = span\{c_i^p - (c_i^*)^p | i \in \Omega, p \geq 0\} + K(I,*)$. The algebra I has an infinite family of pairwise orthogonal idempotents, which includes vertices from W. Choose $w \in W$. Then $K(I,*) = wK(I,*)w + [K(I,*), K(I,*)]$.
Now, $[c_i^p - (c_i^*)^p, K(I,*)] = [c_i^p - (c_i^*)^p, wK(I,*)w + [K(I,*), K(I,*)]] = [c_i^p - (c_i^*)^p, [K(I,*), K(I,*)]]$ which lies in $[K(I,*), K(I,*)]$. Hence $[K(L(\Gamma),*), K(I,*)] \subseteq [K(I,*), K(I,*)]$. Let us analyze an element $[c_i^p - (c_i^*)^p, c_i^q - (c_i^*)^q]$ which is equal to $-[c_i^p + (c_i^*)^p, c_i^q + (c_i^*)^q]$. Furthermore, $c_i^p + (c_i^*)^p = (c_i + c_i^*)^p + \sum_{k<p} \alpha_k (c_i^k + (c_i^*)^k) + h, \quad h \in H(I,*) = \{a \in I | a^* = a\}$. This implies that $[c_i^p - (c_i^*)^p, c_i^q - (c_i^*)^q] \in [H(L(\Gamma),*), H(I,*)]$. I. Herstein proved that for a simple algebra A of characteristic not equal to 2 and of dimension not equal to 1 or 4 we have $H(A,*) = span\{k^2 | k \in K(A,*)\}$. The algebra I has an infinite family of pairwise orthogonal idempotents and therefore is infinite dimensional. If $h \in H(L(\Gamma),*), k \in K(I,*)$ then $[h, k^2] = [hk + kh, k] \in [K(L(\Gamma),*), K(I,*)] = [K(I,*), K(I,*)]$. Thus $[K(L(\Gamma),*), K(L(\Gamma),*)] = [K(I,*), K(I,*)]$. The latter algebra is simple, again by the theorem of I. Herstein. □

Acknowledgement

The authors would like to thank professor Efim Zelmanov for his constant advise and valuable help during the preparation of this work. The authors would also like to express their appreciation to professor S. K. Jain for carefully reading the manuscript and for offering his comments. This paper was funded by King Abdulaziz University, under grant No. (7-130/1433 HiCi). The authors, therefore, acknowledge technical and financial support of KAU.

References

1. G. Abrams, G. Aranda Pino, *The Leavitt path algebra of a graph,* J. Algebra 293 (2005), 319-334
2. G. Abrams, Z. Mesyan, *Simple Lie algebra arising from Leavitt path algebra,* Journal of pure and applied algebra 216, (2012), 2303-2313.

3. A. Alahmadi, H. Alsulami, S.K. Jain, E. Zelmanov, *Leavitt path algebras of finite Gelfand Kirillov dimension,* Journal of Algebra and Its Applications, 171(2012)
4. I.N.Herstein, *Topics in Ring Theory,* Mathematics Lecture Notes, University of Chicago,1965
5. I.N.Herstein, *Rings with Involution,* Mathematics Lecture Notes, University of Chicago,1976.

Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah, 21589, Saudi Arabia

Current address: Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah, 21589, Saudi Arabia

E-mail address: adelnife2@yahoo.com

Department of Mathematics, King Abdulaziz University, P.O.Box 80203, Jeddah, 21589, Saudi Arabia

E-mail address: hhaalsalmi@kau.edu.sa